

# Functions of locally bounded variation on Wiener spaces

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We introduce the concept of functions of locally bounded variation on abstract Wiener spaces and study their properties. Some nontrivial examples and applications to stochastic analysis are also discussed.

## 1. Introduction

Functions of bounded variation (BV functions) on abstract Wiener spaces were first studied in [5; 6] for their applications in stochastic analysis, but recently BV functions on infinite dimensional spaces have attracted attention for a variety of reasons. In this paper, we newly introduce the space  $\dot{B}V_{\text{loc}}$  of functions of locally bounded variation (local BV functions) on the abstract Wiener space  $(E, H, \mu)$ . Needless to say, there could be several ways of localizing the concept of bounded variation. In this article, we adopt the ideas in the theory of (quasi-)regular Dirichlet forms, which are suitable for application to stochastic analysis. Indeed, we show that a Dirichlet form of type

$$\mathcal{E}^\rho(f, g) = \frac{1}{2} \int_E \langle \nabla f, \nabla g \rangle_H \rho \, d\mu$$

associated with a nonnegative function  $\rho$  in  $\dot{B}V_{\text{loc}}$  with some extra assumptions provides a diffusion process that has the Skorokhod representation (Theorem 10). This result is regarded as a natural generalization of [6, Theorem 4.2], where  $\rho$  is assumed to be a BV function instead. We also consider the classical Wiener space on  $\mathbb{R}^d$  as  $E$  and provide a sufficient condition for an open set  $O$  of  $\mathbb{R}^d$  so that the indicator function of the set of all paths staying in the closure  $\overline{O}$  is a local BV function. Accordingly, we can construct the (modified) reflecting Ornstein–Uhlenbeck process with

the Skorokhod representation on the set of paths staying in  $\overline{O}$  under a rather weak condition on  $O$ . This is a study related to another paper [9], in which a sufficient condition was given for the above-mentioned indicator function to be a BV function on either pinned path spaces or one-sided pinned spaces.

The remainder of this paper is organized as follows. In Section 2, we introduce the concept of local BV functions on a Wiener space and study their properties. In Section 3, we provide a sufficient condition for a class of indicator functions to be local BV functions.

## 2. The space $BV_{\text{loc}}$ on Wiener spaces

Let  $(E, H, \mu)$  be an abstract Wiener space. That is,  $E$  is a separable Banach space,  $H$  is a separable Hilbert space densely and continuously embedded in  $E$ , and  $\mu$  is a Gaussian measure on  $E$  that satisfies

$$\int_E \exp(\sqrt{-1}l(z)) \mu(dz) = \exp(-|l|_H^2/2), \quad l \in E^*.$$

Here, we regard the topological dual of  $E$ ,  $E^*$ , as a dense subspace of  $H$  by the natural identification  $H^* \simeq H$ . The inner product and the norm of  $H$  are denoted by  $\langle \cdot, \cdot \rangle_H$  and  $|\cdot|_H$ , respectively. We set

$$\mathcal{FC}_b^1 = \left\{ u: E \rightarrow \mathbb{R} \left| \begin{array}{l} u(z) = f(l_1(z), \dots, l_m(z)), \quad l_1, \dots, l_m \in E^*, \\ f \in C_b^1(\mathbb{R}^m) \text{ for some } m \in \mathbb{N} \end{array} \right. \right\}$$

and define  $\mathcal{FC}_b^1(E^*)$  as the set of all linear combinations of  $H$ -valued functions  $u(\cdot)l$  on  $E$  with  $u \in \mathcal{FC}_b^1$  and  $l \in E^* \subset H$ . For  $u \in \mathcal{FC}_b^1$ ,  $\nabla u$  denotes the  $H$ -derivative of  $u$ , which is an  $H$ -valued function on  $E$  that is characterized by the identity

$$\langle \nabla u(z), l \rangle_H = \lim_{\varepsilon \rightarrow 0} (u(z + \varepsilon l) - u(z))/\varepsilon, \quad l \in E^* \subset H \subset E.$$

In the same way,  $\nabla u$  is defined for  $u \in \mathcal{FC}_b^1(E^*)$  as an  $H \otimes H$ -valued function on  $E$ .

Let  $(X, \mathcal{X}, \nu)$  be a measure space and let  $Y$  be a separable Hilbert space with norm  $|\cdot|_Y$ . For  $p \in [1, +\infty]$ , we denote by  $L^p(X \rightarrow Y; \nu)$  the space of all  $Y$ -valued  $L^p$ -functions on  $(X, \mathcal{X}, \nu)$  with the norm defined as

$$\|f\|_{L^p(\nu)} = \begin{cases} \left( \int_X |f(x)|_Y^p \nu(dx) \right)^{1/p}, & p \in [1, +\infty), \\ \nu\text{-ess sup}_{x \in E} |f(x)|_Y, & p = +\infty. \end{cases}$$

As usual, two functions that are equal a.e. are identified. If  $X = E$  or  $\nu = \mu$  or  $Y = \mathbb{R}$ , we often omit these symbols from the notation. In particular,  $L^p$  and  $\|\cdot\|_p$  represent  $L^p(E; \mu)$  and  $\|\cdot\|_{L^p(\mu)}$ , respectively. For  $p \in [1, +\infty)$ , the closure of  $\mathcal{FC}_b^1$  (resp.  $\mathcal{FC}_b^1(E^*)$ ) with respect to the norm  $(\|\cdot\|_p^p + \|\nabla \cdot\|_p^p)^{1/p}$  will be denoted by  $\mathbb{D}^{1,p}$  (resp.  $\mathbb{D}^{1,p}(H)$ ). The operator  $\nabla$  extends to a continuous map from  $\mathbb{D}^{1,p}$  to  $L^p(E \rightarrow H)$ . Let  $\nabla^*$  denote the adjoint operator of  $\nabla$ . This is regarded as a bounded operator from  $\mathbb{D}^{1,p}(H)$  to  $L^p$  for  $p \in (1, \infty)$ . We note that both  $\nabla$  and  $\nabla^*$  have the local property in the sense that, if  $f \in \mathbb{D}^{1,2}$  (resp.  $G \in \mathbb{D}^{1,2}(H)$ ) satisfies  $f = 0$  (resp.  $G = 0$ )  $\mu$ -a.e. on some measurable set  $A$ , then  $\nabla f = 0$  (resp.  $\nabla^* G = 0$ )  $\mu$ -a.e. on  $A$ . See, e.g., [11, Propositions 1.3.16 and 1.3.15] for the proof.

Let  $L(\log L)^{1/2}$  denote the set of all real-valued  $\mu$ -measurable functions  $f$  on  $E$  such that  $f(0 \vee \log |f|)^{1/2} \in L^1$ . For  $\rho \in L(\log L)^{1/2}$ , define

$$V(\rho) = \sup \left\{ \int_E (\nabla^* G) \rho d\mu \left| G \in \mathcal{FC}_b^1(E^*), |G(z)|_H \leq 1 \text{ for every } z \in E \right. \right\} \\ (\leq +\infty).$$

The function space  $BV$  on  $E$  is defined as

$$BV = \{\rho \in L(\log L)^{1/2} \mid V(\rho) < \infty\}.$$

A function in  $BV$  is called a BV function or a function of bounded variation. We remark that the following inequality holds for  $p \in (1, +\infty]$  and  $\rho \in BV \cap L^p$ .

$$\int_E (\nabla^* G) \rho d\mu \leq V(\rho) \|G\|_\infty, \quad G \in \mathbb{D}^{1,q}(H), \quad (1)$$

where  $q$  is the conjugate exponent of  $p$ .

For a function space  $\mathcal{C}$  on  $E$  and a  $\mu$ -measurable subset  $A$  of  $E$ ,  $\mathcal{C}_A$  denotes the space of all functions in  $\mathcal{C}$  vanishing  $\mu$ -a.e. on  $E \setminus A$ , and  $\mathcal{C}_b$  denotes the set of all bounded functions in  $\mathcal{C}$ . Moreover,  $\mathcal{C}_{A,b}$  denotes  $\mathcal{C}_A \cap \mathcal{C}_b$ .

For  $\xi \in L^1$ ,  $F^\xi$  denotes the support of the measure  $|\xi| \cdot \mu$ . For  $\xi \in L^1$  with  $\xi \geq 0$   $\mu$ -a.e., we define a bilinear form  $(\mathcal{E}, \mathcal{FC}_b^1)$  on  $L^2(F^\xi; \xi \cdot \mu)$  by

$$\mathcal{E}^\xi(f, g) = \frac{1}{2} \int_E \langle \nabla f, \nabla g \rangle_H \xi d\mu, \quad f, g \in \mathcal{FC}_b^1.$$

The set of all  $\xi$  such that  $(\mathcal{E}, \mathcal{FC}_b^1)$  is closable on  $L^2(F^\xi; \xi \cdot \mu)$  is denoted by  $QR$ . For  $\xi \in QR$ , the closure of  $(\mathcal{E}, \mathcal{FC}_b^1)$ , denoted by  $(\mathcal{E}^\xi, \mathcal{F}^\xi)$ , is a quasi-regular Dirichlet form on  $L^2(F^\xi; \xi \cdot \mu)$  (see, e.g., [12] and [5, Theorem 2.1])

for the proof). The space  $\mathcal{F}^\xi$  is regarded as a Hilbert space by using the inner product  $(f, g) \mapsto \mathcal{E}^\xi(f, g) + \int_{F^\xi} fg \xi d\mu$ . The associated capacity  $\text{Cap}^\xi$  is then defined as

$$\text{Cap}^\xi(A) = \inf \left\{ \mathcal{E}^\xi(f, f) + \|f\|_{L^2(\xi \cdot \mu)}^2 \left| \begin{array}{l} f \in \mathcal{F}^\xi \text{ and } f \geq 1 \text{ } \xi \cdot \mu\text{-a.e. on} \\ \text{some open set including } A \end{array} \right. \right\}$$

for  $A \subset F^\xi$ . The concept “ $\mathcal{E}^\xi$ -quasi-everywhere” ( $\mathcal{E}^\xi$ -q.e.) is based on this capacity. An increasing sequence  $\{F_k\}_{k=1}^\infty$  of closed sets in  $F^\xi$  is called an  $\mathcal{E}^\xi$ -nest if  $\lim_{k \rightarrow \infty} \text{Cap}^\xi(F^\xi \setminus F_k) = 0$ . It is known that  $\{F_k\}_{k=1}^\infty$  is an  $\mathcal{E}^\xi$ -nest if and only if  $\bigcup_{k=1}^\infty \mathcal{F}_{F_k}^\xi$  is dense in  $\mathcal{F}^\xi$ . In this situation, the set  $S^\xi$  of all smooth measures is described as the totality of all positive Borel measures  $\nu$  on  $F^\xi$  such that  $\nu$  charges no set of zero capacity  $\text{Cap}^\xi$  and there exists an  $\mathcal{E}^\xi$ -nest  $\{F_k\}_{k=1}^\infty$  such that  $\nu(F_k) < \infty$  for all  $k$ . A function  $f$  on  $E$  is called  $\mathcal{E}^\xi$ -quasi-continuous if there exists an  $\mathcal{E}^\xi$ -nest  $\{F_k\}_{k=1}^\infty$  such that  $f|_{F_k}$  is continuous on  $F_k$  for every  $k$ . Any function  $f \in \mathcal{F}^\xi$  has an  $\mathcal{E}^\xi$ -quasi-continuous modification  $\tilde{f}$ . When  $\xi \equiv 1$ , we write  $\mathcal{E}$ ,  $\mathcal{F}$ , and  $\text{Cap}$  instead of  $\mathcal{E}^\xi$ ,  $\mathcal{F}^\xi$ , and  $\text{Cap}^\xi$ , respectively. We note that  $\mathcal{F} = \mathbb{D}^{1,2}$ . For a subset  $A$  of  $E$ ,  $e_A$  denotes the equilibrium potential of  $A$  with respect to  $(\mathcal{E}, \mathcal{F})$ : that is,  $e_A$  attains the infimum of  $\{\mathcal{E}(f, f) + \|f\|_2^2 \mid f \in \mathbb{D}^{1,2} \text{ and } \tilde{f} \geq 1 \text{ } \mathcal{E}\text{-q.e. on } A\}$ , which is equal to  $\text{Cap}(A)$  (cf. [7, Theorem 2.1.5]). Henceforth, we will always assume that  $e_A$  is  $\mathcal{E}$ -quasi-continuous by itself. Any  $G \in \mathbb{D}^{1,2}(H)$  also has an  $\mathcal{E}$ -quasi-continuous modification  $\tilde{G}$  (cf. [2, Chapter VII, Theorem 1.3.1]).

One of the fundamental properties of BV functions is given by the following theorem.

**Theorem 1** (cf. [6, Theorem 3.9]). *For  $\rho \in BV$ , there exists a positive finite measure  $\nu$  on  $E$  and an  $H$ -valued Borel function  $\sigma$  on  $E$  such that  $|\sigma|_H = 1$   $\nu$ -a.e., and for every  $G \in \mathcal{FC}_b^1(E^*)$ ,*

$$\int_E (\nabla^* G) \rho d\mu = \int_E \langle G, \sigma \rangle_H d\nu. \quad (2)$$

*The measure  $\nu$  belongs to  $S^{|\rho|+1}$ . If  $\rho \in QR$  in addition, then  $\nu|_{E \setminus F^\rho} = 0$  and  $\nu|_{F^\rho} \in S^\rho$ . Also,  $\nu$  and  $\sigma$  are uniquely determined in the following sense: if  $\nu'$  and  $\sigma'$  are another pair satisfying (2) for all  $G \in \mathcal{FC}_b^1(E^*)$ , then  $\nu = \nu'$  and  $\sigma = \sigma'$   $\nu$ -a.e.*

We now introduce some localized function spaces.

**Definition 2.** Let  $\rho$  be a real-valued  $\mu$ -measurable function on  $E$ .

- (i) For  $p \in [1, +\infty]$ , we say  $\rho \in \dot{L}_{\text{loc}}^p$  if there exists an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  such that  $\rho \cdot \mathbf{1}_{F_k} \in L^p$  for every  $k \in \mathbb{N}$ .
- (ii) We say  $\rho \in \dot{BV}_{\text{loc}}$  if there exists an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  and  $\{\rho_k\}_{k=1}^\infty \subset BV$  such that  $\rho = \rho_k$   $\mu$ -a.e. on  $F_k$  for every  $k \in \mathbb{N}$ .

The following are some properties of  $\dot{BV}_{\text{loc}}$ .

**Theorem 3.** *For  $\rho \in \dot{L}_{\text{loc}}^2$ , the implications (a)  $\Leftrightarrow$  (b)  $\Leftarrow$  (c) hold.*

- (a)  $\rho \in \dot{BV}_{\text{loc}}$ .
- (b) *There exists an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  and positive numbers  $\{c_k\}_{k=1}^\infty$  such that for every  $k \in \mathbb{N}$ ,  $\rho \cdot \mathbf{1}_{F_k} \in L^2$  and*

$$\int_E \rho \nabla^* G d\mu \leq c_k \|G\|_\infty, \quad G \in \mathbb{D}^{1,2}(H)_{F_k, b}.$$

- (c) *There exists an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  and  $\{f_m\}_{m=1}^\infty \subset \mathbb{D}^{1,1}$  such that for every  $k \in \mathbb{N}$ ,  $(f_m - \rho) \cdot \mathbf{1}_{F_k}$  converges to 0 in  $L^1$  as  $m \rightarrow \infty$  and  $\|(\nabla f_m) \cdot \mathbf{1}_{F_k}\|_1$  is bounded in  $m$ .*

Unlike the case of finite-dimensional spaces, we must take care with the localized test functions on  $E$ . Indeed, except for the zero function there are no functions  $f$  in  $\mathcal{FC}_b^1$  such that  $f$  vanishes outside a bounded set. This is why we introduce  $\mathbb{D}^{1,2}(H)_{F_k, b}$  in the theorem above.

For the proof of Theorem 3, we introduce some lemmas.

**Lemma 4.** *Let  $\{F_k\}_{k=1}^\infty$  be an  $\mathcal{E}$ -nest. Then, there exists another  $\mathcal{E}$ -nest  $\{F'_k\}_{k=1}^\infty$  and functions  $\{\varphi_k\}_{k=1}^\infty \subset \bigcup_{n=1}^\infty \mathbb{D}_{F_n}^{1,2}$  such that for any  $k$ ,  $F'_k \subset F_l$  for some  $l \geq k$ ,  $\varphi_k$  is  $\mathcal{E}$ -quasi-continuous,  $0 \leq \varphi_k \leq 1$  on  $E$ ,  $\varphi_k = 1$  on  $F'_k$ , and  $\lim_{k \rightarrow \infty} \mathcal{E}(\varphi_k, \varphi_k) = 0$ .*

**Proof.** We denote  $E \setminus F_k$  by  $F_k^c$ , etc. Take an increasing sequence  $\{n(k)\}_{k=1}^\infty$  of natural numbers such that  $\text{Cap}(F_{n(k)}^c) < 2^{-k}$  for every  $k$ . There exists an  $\mathcal{E}$ -nest  $\{\hat{F}_l\}_{l=1}^\infty$  such that  $e_{F_{n(k)}^c}$  is continuous on each  $\hat{F}_l$  for every  $k$  (cf. [7, Theorem 2.1.2]). For  $k \in \mathbb{N}$ , let  $\varphi_k = 1 \wedge 2(1 - e_{F_{n(k)}^c})$  and  $\tilde{F}_k = \{\varphi_k = 1\}$ . Since  $\tilde{F}_k^c = \{e_{F_{n(k)}^c} > 1/2\}$ ,

$$\text{Cap}(\tilde{F}_k^c) \leq \mathcal{E}(2e_{F_{n(k)}^c}, 2e_{F_{n(k)}^c}) + \|2e_{F_{n(k)}^c}\|_2^2 = 4 \text{Cap}(F_{n(k)}^c) < 2^{-k+2}.$$

Define  $F'_k = \bigcap_{l=k}^\infty (\tilde{F}_l \cap \hat{F}_l \cap F_{n(l)})$ . Then,  $\{F'_k\}_{k=1}^\infty$  is a nondecreasing

sequence of closed sets and

$$\begin{aligned}
\text{Cap}((F'_k)^c) &= \text{Cap}\left(\left(\bigcup_{l=k}^{\infty} \check{F}_l^c\right) \cup \hat{F}_k^c \cup F_{n(k)}^c\right) \\
&\leq \sum_{l=k}^{\infty} \text{Cap}(\check{F}_l^c) + \text{Cap}(\hat{F}_k^c) + \text{Cap}(F_{n(k)}^c) \\
&\leq 2^{-k+3} + \text{Cap}(\hat{F}_k^c) + \text{Cap}(F_{n(k)}^c),
\end{aligned}$$

which converges to 0 as  $k \rightarrow \infty$ . Furthermore,  $\mathcal{E}(\varphi_k, \varphi_k) \leq 4\mathcal{E}(e_{F_{n(k)}^c}, e_{F_{n(k)}^c}) < 2^{-k+2} \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

**Lemma 5.** *Let  $p \in (1, +\infty]$  and  $q$  denote the conjugate exponent of  $p$ .*

- (i) *Let  $f \in BV \cap L^p$  and  $g \in \mathbb{D}_b^{1,q}$ . Then,  $fg \in BV$  and  $V(fg) \leq V(f)\|g\|_{\infty} + \|f\nabla g\|_1$ .*
- (ii) *Let  $f \in BV$  and  $g \in \mathbb{D}_{F,b}^{1,q}$  for some  $\mu$ -measurable set  $F \subset E$ . If  $f \cdot \mathbf{1}_F \in L^p$ , then  $fg \in BV \cap L^p$  and  $V(fg) \leq V(f)\|g\|_{\infty} + \|f\nabla g\|_1$ .*

**Proof.** (i) Let  $\{T_t\}_{t>0}$  be the Ornstein–Uhlenbeck semigroup on  $E$ . For  $t > 0$ ,  $T_t f \in \mathbb{D}^{1,p}$ , thus  $(T_t f)g \in \mathbb{D}^{1,1}$  and

$$\begin{aligned}
\|\nabla((T_t f)g)\|_1 &\leq \|\nabla(T_t f)\|_1 \|g\|_{\infty} + \|(T_t f)\nabla g\|_1 \\
&\leq e^{-t} V(f) \|g\|_{\infty} + \|(T_t f)\nabla g\|_1,
\end{aligned}$$

from, e.g., [6, Proposition 3.6]. Since  $\lim_{t \rightarrow 0} (T_t f)g = fg$  in  $L^1$  and  $\lim_{t \rightarrow 0} (T_t f)\nabla g = f\nabla g$  in  $L^1(H)$ , Theorem 3.7 of [6] completes the proof.

(ii) For  $n \in \mathbb{N}$ , let  $f_n = (-n) \vee (f \wedge n)$ . From [6, Corollary 3.8],  $f_n \in BV$  and  $V(f_n) \leq V(f)$ . By the assumption and (i),  $f_n g \in BV \cap L^p$ ,  $\|f_n g\|_p \leq \|fg\|_p$ , and

$$V(f_n g) \leq V(f_n) \|g\|_{\infty} + \|f_n \nabla g\|_1 \leq V(f) \|g\|_{\infty} + \|f \nabla g\|_1,$$

which is bounded for each  $n$ . Since  $\lim_{n \rightarrow \infty} f_n g = fg$  in  $L^1$ , the claim follows.  $\square$

**Lemma 6.** *For  $\rho \in \dot{BV}_{\text{loc}} \cap \dot{L}^2$ , there exists an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^{\infty}$  and functions  $\{\rho_k\}_{k=1}^{\infty}$  in  $BV \cap L^2$  such that  $\rho = \rho_k$  on  $F_k$  for every  $k \in \mathbb{N}$ .*

**Proof.** There exist an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^{\infty}$  and  $\{\rho_k\}_{k=1}^{\infty} \subset BV$  such that for every  $k$ ,  $\rho \cdot \mathbf{1}_{F_k} \in L^2$  and  $\rho = \rho_k$  on  $F_k$ . Take the  $\mathcal{E}$ -nest  $\{F'_k\}_{k=1}^{\infty}$  and the functions  $\{\varphi_k\}_{k=1}^{\infty}$  in Lemma 4. From Lemma 5, the assertion is true by taking  $\{F'_k\}_{k=1}^{\infty}$  and  $\{\rho \varphi_k\}_{k=1}^{\infty}$  as  $\{F_k\}_{k=1}^{\infty}$  and  $\{\rho_k\}_{k=1}^{\infty}$ , respectively.  $\square$

**Proof of Theorem 3.** (a) $\Rightarrow$ (b): Take the  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  and  $\{\rho_k\}_{k=1}^\infty \subset BV \cap L^2$  as in Lemma 6. Let  $k \in \mathbb{N}$  and take  $G \in \mathbb{D}^{1,2}(H)_{F_k,b}$ . Since  $\nabla^* G = 0$   $\mu$ -a.e. on  $E \setminus F_k$ ,

$$\int_E \rho \nabla^* G d\mu = \int_E \rho_k \nabla^* G d\mu \leq V(\rho_k) \|G\|_\infty,$$

where the last inequality follows from (1). Thus, (b) holds.

(b) $\Rightarrow$ (a): Take the  $\mathcal{E}$ -nest  $\{F'_k\}_{k=1}^\infty$  and the functions  $\{\varphi_k\}_{k=1}^\infty$  as in Lemma 4. For  $k \in \mathbb{N}$ , take  $l \in \mathbb{N}$  such that  $\varphi_k \in \mathbb{D}_{F'_l}^{1,2}$  and let  $\rho_k = \rho \varphi_k$ . Then,  $\rho_k \in L^2$  and  $\rho_k = \rho$  on  $F'_k$ . Moreover, for any  $G \in \mathcal{FC}_b^1(E^*)$ ,

$$\begin{aligned} \int_E \rho_k \nabla^* G d\mu &= \int_E \rho \varphi_k \nabla^* G d\mu \\ &= \int_E \rho \{ \nabla^*(\varphi_k G) + \langle \nabla \varphi_k, G \rangle_H \} d\mu \\ &\leq c_l \|\varphi_k G\|_\infty + \|\rho \cdot \mathbf{1}_{F'_l}\|_2 \|\nabla \varphi_k\|_2 \|G\|_\infty \\ &\leq (c_l + \|\rho \cdot \mathbf{1}_{F'_l}\|_2 \|\nabla \varphi_k\|_2) \|G\|_\infty. \end{aligned}$$

Therefore,  $\rho_k \in BV$ .

(c) $\Rightarrow$ (b): We may assume that  $\rho \cdot \mathbf{1}_{F_k} \in L^2$  for every  $k$ . For  $M > 0$ , define  $\Phi_M(t) = (-M) \vee (t \wedge M)$  for  $t \in \mathbb{R}$ . Then, for  $k \in \mathbb{N}$  and  $G \in \mathbb{D}^{1,2}(H)_{F_k,b}$ ,

$$\begin{aligned} \int_E \Phi_M(f_m) \nabla^* G d\mu &= \int_E \langle \nabla(\Phi_M(f_m)), G \rangle_H d\mu \\ &\leq \|(\nabla f_m) \cdot \mathbf{1}_{F_k}\|_1 \|G\|_\infty \leq c_k \|G\|_\infty, \end{aligned}$$

where  $c_k := \sup_{m \in \mathbb{N}} \|(\nabla f_m) \cdot \mathbf{1}_{F_k}\|_1 < \infty$ . Since  $\nabla^* G \in L^2$ ,

$$\lim_{m \rightarrow \infty} \int_E \Phi_M(f_m) \nabla^* G d\mu = \int_E \Phi_M(\rho) \nabla^* G d\mu.$$

Since  $\lim_{M \rightarrow \infty} (\Phi_M(\rho) - \rho) \cdot \mathbf{1}_{F_k} = 0$  in  $L^2$ ,

$$\lim_{M \rightarrow \infty} \int_E \Phi_M(\rho) \nabla^* G d\mu = \int_E \rho \nabla^* G d\mu.$$

Therefore, we have  $\int_E \rho \nabla^* G d\mu \leq c_k \|G\|_\infty$ .  $\square$

The following are some basic lemmas that are used later.

**Lemma 7.** *The following claims hold.*

(i) *For  $\rho \in L^\infty$ ,  $\mathcal{F}^{|\rho|+1} = \mathbb{D}^{1,2}$  and their norms are equivalent.*

(ii) Let  $\rho \in QR \cap L^\infty$  and  $\{F_k\}_{k=1}^\infty$  be an  $\mathcal{E}$ -nest. Then,  $\mathbb{D}^{1,2}|_{F^\rho}$  is continuously embedded in  $\mathcal{F}^\rho$  and there exists  $c > 0$  such that  $\text{Cap}^\rho(A) \leq c \text{Cap}(A)$  for any  $A \subset F^\rho$ . In particular,  $\{F_k \cap F^\rho\}_{k=1}^\infty$  is an  $\mathcal{E}^\rho$ -nest. Moreover,  $\bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}|_{F^\rho}$  is dense in  $\mathcal{F}^\rho$ .

**Proof.** We prove only the last claim of (ii). It suffices to prove that for any function  $f \in \mathcal{F}C_b^1$ ,  $f|_{F^\rho}$  can be approximated by elements of  $\bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}|_{F^\rho}$  in  $\mathcal{F}^\rho$ . Since  $\bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}$  is dense in  $\mathbb{D}^{1,2}$ , there exists  $\{f_n\}_{n=1}^\infty$  in  $\bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}$  such that  $f_n$  converges to  $f$  in  $\mathbb{D}^{1,2}$ . Then,  $f_n|_{F^\rho}$  converges to  $f|_{F^\rho}$  in  $\mathcal{F}^\rho$ .  $\square$

**Lemma 8.** Let  $\{F_k\}_{k=1}^\infty$  be an  $\mathcal{E}$ -nest. Let  $\nu$  and  $\nu'$  be  $\mathcal{E}$ -smooth measures on  $E$  such that  $\nu(F_k) < \infty$  and  $\nu'(F_k) < \infty$  for every  $k \in \mathbb{N}$ . Let  $\sigma$  and  $\sigma'$  be  $H$ -valued Borel functions on  $E$  such that  $|\sigma|_H = 1$   $\nu$ -a.e. and  $|\sigma'|_H = 1$   $\nu'$ -a.e. If  $\int_E \langle \tilde{G}, \sigma \rangle_H d\nu = \int_E \langle \tilde{G}, \sigma' \rangle_H d\nu'$  for every  $G \in \bigcup_{k=1}^\infty \mathbb{D}^{1,2}(H)_{F_k, b}$ , then  $\nu = \nu'$  and  $\sigma = \sigma'$   $\nu$ -a.e.

**Proof.** Let  $\xi = \nu + \nu'$  and  $\gamma = \sigma \frac{d\nu}{d\xi} - \sigma' \frac{d\nu'}{d\xi}$ . Then,  $\int_E \langle \tilde{G}, \gamma \rangle_H d\xi = 0$  for every  $G \in \bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}$ . Taking a uniformly bounded sequence  $\{G_n\}_{n=1}^\infty$  from  $\bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}$  such that  $\langle \tilde{G}_n, \gamma \rangle_H \rightarrow |\gamma|_H$   $\xi$ -a.e. as  $n \rightarrow \infty$ , we obtain  $\gamma = 0$   $\xi$ -a.e. Therefore,  $|\sigma|_H \frac{d\nu}{d\xi} = |\sigma'|_H \frac{d\nu'}{d\xi}$   $\xi$ -a.e. Since  $|\sigma|_H = 1$   $\nu$ -a.e.,  $|\sigma|_H \frac{d\nu}{d\xi} = \frac{d\nu}{d\xi}$   $\xi$ -a.e. Similarly,  $|\sigma'|_H \frac{d\nu'}{d\xi} = \frac{d\nu'}{d\xi}$   $\xi$ -a.e. Then,  $\frac{d\nu}{d\xi} = \frac{d\nu'}{d\xi}$   $\xi$ -a.e., which implies  $\nu = \nu'$ . The identity  $\sigma = \sigma'$   $\nu$ -a.e. follows from  $\gamma = 0$   $\xi$ -a.e. and  $\nu = \nu'$ .  $\square$

**Theorem 9.** For  $\rho \in BV_{\text{loc}} \cap \dot{L}^2$ , there exist an  $H$ -valued Borel function  $\sigma$  on  $E$ , an  $\mathcal{E}$ -smooth measure  $\nu$  on  $E$ , and an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  such that  $\rho \cdot \mathbf{1}_{F_k} \in L^2$  and  $\nu(F_k) < \infty$  for every  $k \in \mathbb{N}$ ,  $|\sigma|_H = 1$   $\nu$ -a.e., and

$$\int_E \rho \nabla^* G d\mu = \int_E \langle \tilde{G}, \sigma \rangle_H d\nu \quad \text{for every } G \in \bigcup_{k=1}^\infty \mathbb{D}^{1,2}(H)_{F_k, b}, \quad (3)$$

where  $\tilde{G}$  denotes an  $\mathcal{E}$ -quasi-continuous modification of  $G$ . The pair  $\nu$  and  $\sigma$  is uniquely determined in the following sense: if another pair  $\nu'$  and  $\sigma'$  with some  $\mathcal{E}$ -nest satisfies the conditions above, then  $\nu = \nu'$  and  $\sigma = \sigma'$   $\nu$ -a.e.

If  $\rho \in QR \cap L^\infty$  in addition, then  $\nu|_{E \setminus F^\rho} = 0$  and  $\nu|_{F^\rho} \in S^\rho$ .

**Proof.** Take an  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  and  $\{\rho_k\}_{k=1}^\infty \subset BV \cap L^2$  as in Lemma 6. From Theorem 1, for each  $k$ , there exist an  $H$ -valued Borel function  $\sigma_k$  on



$E$  and an  $\mathcal{E}$ -smooth finite measure  $\nu_k$  on  $E$  such that  $|\sigma_k|_H = 1$   $\nu_k$ -a.e. and

$$\int_E \rho_k \nabla^* G d\mu = \int_E \langle G, \sigma_k \rangle_H d\nu_k \quad (4)$$

for every  $G \in \mathcal{FC}_b^1(E^*)$ . By approximation, (4) holds for  $G \in \mathbb{D}^{1,2}(H)_b$  with  $\langle G, \sigma_k \rangle_H$  replaced by  $\langle \tilde{G}, \sigma_k \rangle_H$ .

Take an  $\mathcal{E}$ -nest  $\{F'_k\}_{k=1}^\infty$  and functions  $\{\varphi_k\}_{k=1}^\infty$  as in Lemma 4. For  $k \in \mathbb{N}$ , let  $n(k)$  denote a number such that  $\varphi_k \in \mathbb{D}_{F_{n(k)}}^{1,2}$ . We may assume that  $\{n(k)\}_{k=1}^\infty$  is an increasing sequence. Then, for  $l \geq k$ ,

$$\varphi_k \cdot \nu_{n(k)} = \varphi_k \cdot \nu_{n(l)} \text{ and } \sigma_{n(k)} = \sigma_{n(l)} \text{ } (\varphi_k \cdot \nu_{n(k)})\text{-a.e.} \quad (5)$$

Indeed, for any  $G \in \mathbb{D}^{1,2}(H)_b$ , we have  $\varphi_k G \in \mathbb{D}^{1,2}(H)_{F_{n(k)},b}$  and

$$\begin{aligned} \int_E \varphi_k \langle \tilde{G}, \sigma_{n(k)} \rangle_H d\nu_{n(k)} &= \int_E \rho_{n(k)} \nabla^* (\varphi_k G) d\mu \\ &= \int_E \rho_{n(l)} \nabla^* (\varphi_k G) d\mu \\ &= \int_E \varphi_k \langle \tilde{G}, \sigma_{n(l)} \rangle_H d\nu_{n(l)}. \end{aligned}$$

Thus, (5) follows from Lemma 8. Therefore, we can define  $\sigma$  and  $\nu$  so that  $\sigma = \sigma_{n(k)}$  on  $F'_k$  and  $\nu|_{F'_k} = \nu_{n(k)}|_{F'_k}$  for  $k \in \mathbb{N}$ , and  $\nu(E \setminus \bigcup_{k=1}^\infty F'_k) = 0$ . Then, the conditions described in the theorem are satisfied with  $\{F_k\}_{k=1}^\infty$  replaced by  $\{F'_k\}_{k=1}^\infty$ .

Suppose that another  $\nu'$ ,  $\sigma'$ , and an  $\mathcal{E}$ -nest  $\{F''_k\}_{k=1}^\infty$  satisfy the required conditions. Applying Lemma 8 to the  $\mathcal{E}$ -nest  $\{F_k \cap F''_k\}_{k=1}^\infty$ , we obtain that  $\nu = \nu'$  and  $\sigma = \sigma'$   $\nu$ -a.e.

Lastly, suppose  $\rho \in QR \cap L^\infty$  in addition. Let  $l \in E^*$  and  $g \in \mathbb{D}_{F_k,b}^{1,2}$  for some  $k \in \mathbb{N}$ . Letting  $G = g(\cdot)l$  in (3), we obtain

$$\int_{F^\rho} (\partial_l g + g \cdot l(\cdot)) \rho d\mu = \int_E \tilde{g} \langle l, \sigma \rangle_H d\nu. \quad (6)$$

Denote the left-hand side of (6) by  $I(g)$ . Then,  $I$  provides a bounded functional on  $\mathcal{F}^\rho$  as well as on  $\mathbb{D}^{1,2}$ . From the observation that  $I(g)$  does not change if  $g$  is replaced by  $(-M) \vee (g \wedge M)$  with  $M = \mu\text{-ess sup } |g \cdot \mathbf{1}_{F_k}|$ , the inequality  $I(g) \leq \|g\|_{F^\rho} \|l\|_{L^\infty(\rho \cdot \mu)} \|l\|_H \nu(F_k)$  holds. Then, from Lemma 7 and [8, Theorem 2.18], there exists a unique  $\nu_l \in S^\rho$  such that  $I(g) = \int_{F^\rho} \tilde{g} d\nu_l$  for all  $g \in \bigcup_{k=1}^\infty \mathbb{D}_{F_k,b}^{1,2}$ . From the uniqueness of the integral representation of  $I(g)$  as a functional on  $\mathbb{D}^{1,2}$ , we conclude that  $(\langle l, \sigma \rangle_H \cdot \nu)|_{E \setminus F^\rho} = 0$  and  $(\langle l, \sigma \rangle_H \cdot \nu)|_{F^\rho} = \nu_l \in S^\rho$ . Because  $l$  is arbitrary, we deduce that  $\nu|_{E \setminus F^\rho} = 0$  and  $\nu|_{F^\rho} \in S^\rho$ .  $\square$

We will denote  $\nu$  and  $\sigma$  in the theorem above by  $\|D\rho\|$  and  $\sigma_\rho$ , respectively.

**Theorem 10 (Skorokhod representation).** *Let  $\rho \in \dot{B}V_{\text{loc}} \cap QR \cap L^\infty$ . Then the sample path of the diffusion process  $\mathbf{M}^\rho = (X_t, \mathcal{M}_t, P_z)$  associated with  $(\mathcal{E}^\rho, \mathcal{F}^\rho)$  admits the following expression as a sum of three  $E$ -valued continuous additive functionals.*

$$\begin{aligned} X_t(\omega) - X_0(\omega) \\ = W_t(\omega) - \frac{1}{2} \int_0^t X_s(\omega) ds + \frac{1}{2} \int_0^t \sigma_\rho(X_s(\omega)) dA_s^{\|D\rho\|}(\omega), \quad t \geq 0. \end{aligned} \quad (7)$$

Here,  $A^{\|D\rho\|}$  is a real-valued positive continuous additive functional associated with  $\|D\rho\|$  via the Revuz correspondence. Moreover, for  $\mathcal{E}^\rho$ -q.e.  $z \in F^\rho$ ,  $\{W_t\}_{t \geq 0}$  is the  $\{\mathcal{M}_t\}$ -Brownian motion on  $E$  under  $P_z$ .

**Proof.** The proof is provided along the same lines as those of [6, Theorem 4.2] and [5, Theorem 3.2] with the use of [4, Theorem 6.1] instead of [4, Theorem 6.2] (or [5, Theorem 2.2]). We define  $\{W_t\}_{t \geq 0}$  so that (7) holds. Then,  $\{W_t\}_{t \geq 0}$  is an  $E$ -valued continuous additive functional with the same defining and exceptional sets as  $A^{\|D\rho\|}$ . Take the  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  in Theorem 9. Let  $l \in E^*$  and consider the identity

$$l(X_t) - l(X_0) = l(W_t) - \frac{1}{2} \int_0^t l(X_s) ds + \frac{1}{2} \int_0^t \langle l, \sigma_\rho(X_s) \rangle_H dA_s^{\|D\rho\|}.$$

We note that  $l(\cdot)|_{F^\rho} \in \mathcal{F}^\rho$ . From (3), the identity

$$\mathcal{E}^\rho(l(\cdot), g) = \frac{1}{2} \int_{F^\rho} gl(\cdot) \rho d\mu - \frac{1}{2} \int_{F^\rho} \tilde{g} \langle l, \sigma_\rho \rangle_H d\|D\rho\|$$

holds for  $g \in \bigcup_{k=1}^\infty \mathbb{D}_{F_k, b}^{1,2}$ . Take a nest  $\{F'_k\}_{k=1}^\infty$  and functions  $\{\varphi_k\}_{k=1}^\infty$  as in Lemma 4. For any  $g \in \mathcal{F}_{F'_k, b}^\rho$  with  $k \in \mathbb{N}$ , there exist  $\{g_n\} \subset \bigcup_{m=1}^\infty \mathbb{D}_{F_m, b}^{1,2}$  such that  $\{g_n\}$  are uniformly bounded and  $g_n|_{F^\rho}$  converges to  $g$  in  $\mathcal{F}^\rho$  as  $n \rightarrow \infty$ . Since  $\mathcal{E}(g_n \varphi_k, g_n \varphi_k)^{1/2} \leq \mathcal{E}(g_n, g_n)^{1/2} \|\varphi_k\|_\infty + \mathcal{E}(\varphi_k, \varphi_k)^{1/2} \|g_n\|_\infty$ ,  $\|g_n \varphi_k\|_2 \leq \|g_n\|_\infty \|\varphi_k\|_2$ , and  $(g_n \varphi_k)|_{F^\rho} \rightarrow g$  in  $L^2(\rho \cdot \mu)$ , the Cesàro means of a certain subsequence of  $\{(g_n \varphi_k)|_{F^\rho}\}_{n=1}^\infty$ , which are all elements of  $\mathbb{D}_{F_l, b}^{1,2}|_{F^\rho}$  for some  $l$ , converges to  $g$  in  $\mathcal{F}^\rho$ . A further suitable subsequence of their  $\mathcal{E}^\rho$ -quasi-continuous modifications converges  $\mathcal{E}^\rho$ -q.e. from [7, Theorem 2.1.4]. Therefore, the above identity holds for  $g \in \bigcup_{k=1}^\infty \mathcal{F}_{F'_k, b}^\rho$ , where  $\tilde{g}$  is interpreted as an  $\mathcal{E}^\rho$ -quasi-continuous modification of  $g$ . From [4, Theorem 6.1], the Fukushima decomposition of  $l(X_t) - l(X_0)$  is given by the sum of  $M_t^l := l(W_t)$  and  $N_t^l := -\frac{1}{2} \int_0^t l(X_s) ds + \frac{1}{2} \int_0^t \langle l, \sigma_\rho(X_s) \rangle_H dA_s^{\|D\rho\|}$ . Moreover, the quadratic

variation of  $\{M_t^l\}$  is equal to  $\{t|l|_H^2\}$ . From [6, Lemma 4.1], we conclude that  $\{W_t\}_{t \geq 0}$  is the  $E$ -valued Brownian motion, which completes the proof.  $\square$

### 3. Indicator functions in $B\dot{V}_{\text{loc}}$

In this section, we provide some nontrivial examples of functions in  $B\dot{V}_{\text{loc}}$ . Let  $d \in \mathbb{N}$  and  $T > 0$ . We consider the classical  $d$ -dimensional Wiener space as  $(E, H, \mu)$ : that is,

$$E = \{w \in C([0, T] \rightarrow \mathbb{R}^d) \mid w(0) = 0\},$$

$$H = \left\{ h \in E \left| h \text{ is absolutely continuous and } \int_0^T |\dot{h}(s)|_{\mathbb{R}^d}^2 ds < \infty \right. \right\},$$

and  $\mu$  is the Wiener measure on  $E$ . For a subset  $A$  of  $\mathbb{R}^d$ ,  $\overline{A}$  (resp.  $\partial A$ ) denotes the closure (resp. boundary) of  $A$ , and  $A^c$  denotes  $\mathbb{R}^d \setminus A$ . Define some subsets of  $E$  as follows:

$$\begin{aligned} \Xi_A &= \{w \in E \mid w(t) \in A \text{ for all } t \in [0, T]\}, \\ \Theta_A &= \{w \in E \mid w(t) \in A \text{ for some } t \in [0, T]\}, \\ \partial\Xi_A &= \left\{ w \in E \left| \begin{array}{l} w(t) \in \overline{A} \text{ for all } t \in [0, T] \text{ and} \\ w(s) \in \partial A \text{ for some } s \in [0, T] \end{array} \right. \right\}, \\ \partial'\Xi_A &= \left\{ w \in E \left| \begin{array}{l} w(t) \in \overline{A} \text{ for all } t \in [0, T] \text{ and } w(s) \in \partial A \\ \text{for some unique } s \in [0, T] \end{array} \right. \right\}. \end{aligned}$$

We note that  $\partial\Xi_A$  is the topological boundary of  $\Xi_A$  in  $E$  with the uniform topology. A sufficient condition for  $\mathbf{1}_{\Xi_A}$  to belong to  $BV$  was given in [13; 9]. Here, we provide a sufficient condition for  $\mathbf{1}_{\Xi_A}$  to belong to  $B\dot{V}_{\text{loc}}$ .

For  $x \in \mathbb{R}^d$  and  $r \geq 0$ , we write  $B(x, r)$  and  $\overline{B}(x, r)$  for  $\{z \in \mathbb{R}^d \mid |z - x|_{\mathbb{R}^d} < r\}$  and  $\{z \in \mathbb{R}^d \mid |z - x|_{\mathbb{R}^d} \leq r\}$ , respectively. Let  $O$  be a proper open subset of  $\mathbb{R}^d$  such that  $0 \in O$ . For  $y \in \partial O$ , we define  $\delta(y) \in [0, +\infty]$  as

$$\delta(y) = \sup\{r \geq 0 \mid \text{there exists } z \in O^c \text{ such that } \overline{O} \cap \overline{B}(z, r) = \{y\}\}.$$

From [9, Theorem 5.1],  $\mathbf{1}_{\Xi_{\overline{O}}} \in BV$  if the uniform exterior ball condition

$$\inf_{y \in \partial O} \delta(y) > 0 \tag{8}$$

holds. In order to describe a weaker condition which ensures that  $\mathbf{1}_{\Xi_{\overline{O}}} \in B\dot{V}_{\text{loc}}$ , we introduce the Riesz (and logarithmic) capacities on  $\mathbb{R}^d$ . For a

Borel set  $A$  of  $\mathbb{R}^d$ , let  $\mathcal{P}(A)$  denote the set of all Borel probability measures on  $A$ . For  $\beta \geq 0$ , the  $\beta$ -capacity  $\text{Cap}_\beta(A)$  of  $A$  is defined as

$$\text{Cap}_\beta(A) = \left( \inf_{\lambda \in \mathcal{P}(A)} \iint_{A \times A} g_\beta(|x - y|_{\mathbb{R}^d}) \lambda(dx) \lambda(dy) \right)^{-1},$$

where  $g_\beta(t) = t^{-\beta}$  for  $\beta > 0$  and  $g_0(t) = \log(t^{-1} \vee e)$ . For  $\beta < 0$ , we define  $\text{Cap}_\beta(A) = 1$  for  $A \neq \emptyset$  and  $\text{Cap}_\beta(\emptyset) = 0$ . These are Choquet capacities.

For  $\beta \geq 0$ , let  $\mathcal{H}^\beta(A)$  denote the  $\beta$ -dimensional Hausdorff measure of  $A \subset \mathbb{R}^d$ . It is known that, if a Borel set  $A$  of  $\mathbb{R}^d$  satisfies  $\mathcal{H}^\beta(A) < \infty$ , then  $\text{Cap}_\beta(A) = 0$  (see, e.g., [3, §1, Theorem 1]).

For  $r > 0$ , a closed subset  $\Sigma_r$  of  $\mathbb{R}^d$  is defined as

$$\Sigma_r = \overline{\{y \in \partial O \mid \delta(y) < r\}}. \quad (9)$$

Denote  $\bigcap_{r>0} \Sigma_r$  by  $\Sigma$ . The following is the main theorem of this section.

**Theorem 11.** *Suppose that*

$$\text{Cap}_{d-4}(\Sigma) = 0. \quad (10)$$

*Then,  $\mathbf{1}_{\Xi_{\overline{O}}} \in \dot{B}V_{\text{loc}} \cap QR$ . In particular, the conclusion of Theorem 10 holds with  $\rho = \mathbf{1}_{\Xi_{\overline{O}}}$ . Moreover,  $\text{Cap}^{\mathbf{1}_{\Xi_{\overline{O}}}}(\partial \Xi_{\overline{O}} \setminus \partial' \Xi_{\overline{O}}) = 0$ .*

**Remark 12.**

- (i) The set  $\Xi_{\overline{O}}$  coincides with the closure of  $\Xi_O$  in  $E$ .
- (ii) If  $d \geq 4$  and there exists a sequence of Borel subsets  $\{B_n\}_{n=1}^\infty$  of  $\mathbb{R}^d$  such that  $\bigcup_{n=1}^\infty B_n = \mathbb{R}^d$  and  $\mathcal{H}^{d-4}(\Sigma \cap B_n) < \infty$  for every  $n \in \mathbb{N}$ , then (10) holds.
- (iii) Since  $\inf_{z \in \partial O} \delta(z) > 0$  implies  $\Sigma = \emptyset$ , the uniform exterior condition (8) implies condition (10).
- (iv) The diffusion  $\{X_t\}$  associated with  $(\mathcal{E}^\rho, \mathcal{F}^\rho)$  for  $\rho = \mathbf{1}_{\Xi_{\overline{O}}}$  is regarded as the modified reflecting Ornstein–Uhlenbeck process on  $\Xi_{\overline{O}}$ ; the term “modified” is added since the domain  $\mathcal{F}^\rho$  is defined as the closure of smooth functions and it is not clear whether  $\mathcal{F}^\rho$  is the maximal domain.
- (v) The fact that  $\text{Cap}^{\mathbf{1}_{\Xi_{\overline{O}}}}(\partial \Xi_{\overline{O}} \setminus \partial' \Xi_{\overline{O}}) = 0$  implies that the measure  $\|D\mathbf{1}_{\Xi_{\overline{O}}}\|$  concentrates on  $\partial' \Xi_{\overline{O}}$ , which means that the process  $\{X_t\}$  reflects only at  $\partial' \Xi_{\overline{O}}$  in view of (7).

**Example 13.** Let  $\varphi: [1, 2) \rightarrow \mathbb{R}_+$  be a convex function such that  $\varphi(1) = 0$  and  $\varphi(t) > 0$  for  $t \in (1, 2)$ . Suppose  $d \geq 4$  and consider an open subset  $O$  of  $\mathbb{R}^d$  that is defined by

$$O = B(0, 2) \setminus \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_d \in [1, 2) \text{ and } \varphi(x_d)^2 \geq x_1^2 + \dots + x_{d-1}^2\}.$$

Then  $\Sigma = \{(0, \dots, 0, 1)\}$  and  $\mathcal{H}^{d-4}(\Sigma) < \infty$ . Therefore,  $\mathbf{1}_{\Xi_{\overline{O}}} \in \dot{BV}_{\text{loc}}$  from Theorem 11 and Remark 12 (ii). It is likely that  $\mathbf{1}_{\Xi_{\overline{O}}} \notin BV$  in general, but I have no proof to offer at the moment.

The capacity  $\text{Cap}_{d-4}$  is involved in Theorem 11 for the following reason. From Theorem 1.1 of [10] and the arguments in Section 7 in that paper, for  $a \in (0, T]$  and  $M > 0$ , there exists a constant  $C > 0$  depending only on  $a$  and  $M$  such that, for any closed set  $A$  in  $\overline{B}(0, M) \subset \mathbb{R}^d$ ,

$$\begin{aligned} C^{-1} \text{Cap}_{d-4}(A) &\leq \mathbf{P}(\{Z_s(t) \in A \text{ for some } (s, t) \in [a, T] \times [a, T]\}) \\ &\leq C \text{Cap}_{d-4}(A), \end{aligned}$$

where  $(\{Z_s\}_{s \geq 0}, \mathbf{P})$  is the Ornstein–Uhlenbeck process on  $E$  with initial distribution  $\mu$ . This implies the following property.

$$\begin{aligned} \text{If a closed subset } A \text{ of } \mathbb{R}^d \setminus \{0\} \text{ satisfies } \text{Cap}_{d-4}(A) = 0, \text{ then} \\ \text{Cap}(\Theta_A) = 0. \end{aligned} \tag{11}$$

In particular, since  $0 \notin \Sigma$ , condition (10) implies

$$\text{Cap}(\Theta_{\Sigma}) = 0. \tag{12}$$

For the proof of Theorem 11, we provide some quantitative estimates as in [13; 9]. We define a Lipschitz continuous function  $q$  on  $\mathbb{R}^d$  by

$$q(x) = \inf_{y \in O^c} |x - y|_{\mathbb{R}^d} - \inf_{y \in O} |x - y|_{\mathbb{R}^d}, \quad x \in \mathbb{R}^d.$$

For  $r \geq 0$ , set  $O_r = \{x \in \mathbb{R}^d \mid q(x) > r\}$ . Note that  $O_0 = O$  and  $\{q(x) \geq 0\} = \overline{O}$ .

Let  $W = C([0, \infty) \rightarrow \mathbb{R}^d)$  and  $\mathcal{B}(W)$  be the Borel  $\sigma$ -field of  $W$ . Let  $\{\hat{P}_x\}_{x \in \mathbb{R}^d}$  be the probability measures on  $W$  such that the coordinate process  $\{\omega_t\}_{t \geq 0}$  is the  $d$ -dimensional Brownian motion starting at  $x$  under  $\hat{P}_x$  for each  $x \in \mathbb{R}^d$ . For  $t \geq 0$ , let  $\hat{\mathcal{F}}_t$  denote the  $\sigma$ -field generated by  $\{\omega_s \mid s \in [0, t]\}$ . For an  $\{\hat{\mathcal{F}}_t\}$ -stopping time  $\tau$ , define  $\hat{\mathcal{F}}_{\tau}$  as  $\{A \in \mathcal{B}(W) \mid A \cap \{\tau \leq t\} \in \hat{\mathcal{F}}_t \text{ for all } t \geq 0\}$ . We denote the integral with respect to  $\hat{P}_x$  by  $\hat{E}_x$ . For  $s > 0$ , the shift operator  $\theta_s: W \rightarrow W$  is defined by  $(\theta_s \omega)_t = \omega_{s+t}$ ,  $t \geq 0$ .

The following claims (Lemma 14–Proposition 17) are slight modifications of those in [9].

**Lemma 14.** *Let  $x \in \overline{O}$  and choose  $y \in \partial O$  such that  $q(x) = |x - y|_{\mathbb{R}^d}$ . Suppose that  $\delta(y) > 0$ . Let  $\delta \in (0, \delta(y))$  and take  $z \in O^c$  such that  $B(z, \delta) \cap$*

$\overline{O} = \{y\}$ . Let  $C_\delta = (d-1)/(2\delta)$  and  $R_t = |\omega_t - z|_{\mathbb{R}^d}$  for  $\omega = \{\omega_t\} \in W$ . Then, for each  $u > 0$ ,

$$\{R_t \geq \delta \text{ for all } t \in [0, u]\} \subset \{R_t \leq q(x) + \delta + C_\delta t + S_t \text{ for all } t \in [0, u]\}$$

up to a  $\hat{P}_x$ -null set. Here,  $S_t$  is the 1-dimensional Brownian motion under  $\hat{P}_x$  starting at 0 that is defined by

$$S_t(\omega) = \sum_{i=1}^d \int_0^t \frac{\omega_s^{(i)} - z^{(i)}}{R_s} d\omega_s^{(i)}, \quad \omega_s = (\omega_s^{(1)}, \dots, \omega_s^{(d)}), \quad z = (z^{(1)}, \dots, z^{(d)}),$$

up to the  $\{\hat{\mathcal{F}}_t\}$ -stopping time  $\inf\{t \geq 0 \mid R_t = 0\}$ .

**Proof.** Define  $\sigma = \inf\{t \geq 0 \mid R_t = 0\}$ . Note that  $R_0 = |x - z|_{\mathbb{R}^d} = q(x) + \delta$   $\hat{P}_x$ -a.e. By virtue of Itô's formula,

$$R_t = q(x) + \delta + \int_0^t \frac{d-1}{2R_s} ds + S_t \quad \text{on } \{t < \sigma\} \quad \hat{P}_x\text{-a.e.}$$

Therefore, the assertion holds.  $\square$

**Proposition 15.** *In the same situation as in Lemma 14, for every  $u > 0$ ,*

$$\hat{P}_x \left[ \inf_{t \in [0, u]} q(\omega_t) \geq 0 \right] \leq \left( \frac{d-1}{\delta(y)} + u^{-1/2} \right) q(x).$$

**Proof.** Take  $\delta \in (0, \delta(y))$ . From Lemma 14,

$$\hat{P}_x [R_t \geq \delta \text{ for all } t \in [0, u]] \leq \hat{P}_x [q(x) + \delta + C_\delta t + S_t \geq \delta \text{ for all } t \in [0, u]].$$

Let  $r > q(x)$  and define  $\eta = \inf\{t \geq 0 \mid C_\delta t + S_t \leq -r\}$ . The law of  $\eta$  under  $\hat{P}_x$  is given by

$$\hat{P}_x [\eta \in dt] = \mathbf{1}_{(0, \infty)}(t) \frac{r}{\sqrt{2\pi t^3}} \exp \left( -\frac{(r + C_\delta t)^2}{2t} \right) dt + (1 - e^{-2C_\delta r}) \delta_\infty(dt),$$

where  $\delta_\infty$  is the delta measure at  $\infty$  (see, e.g., [1, p. 295]). Then, we have

$$\begin{aligned} \hat{P}_x \left[ \inf_{t \in [0, u]} q(\omega_t) \geq 0 \right] &\leq \hat{P}_x [R_t \geq \delta \text{ for all } t \in [0, u]] \leq \hat{P}_x [\eta > u] \\ &= \int_u^\infty \frac{r}{\sqrt{2\pi t^3}} \exp \left( -\frac{(r + C_\delta t)^2}{2t} \right) dt + 1 - e^{-2C_\delta r} \\ &\leq \int_u^\infty \frac{r}{\sqrt{2\pi t^3}} dt + 2C_\delta r = \sqrt{\frac{2}{\pi}} \frac{r}{\sqrt{u}} + 2C_\delta r. \end{aligned}$$

Letting  $r \rightarrow q(x)$  and  $\delta \rightarrow \delta(y)$ , we obtain the claim.  $\square$

For  $r > 0$ , define an  $\{\hat{\mathcal{F}}_t\}$ -stopping time  $\tau_r$  by  $\tau_r = \inf\{t \geq 0 \mid \omega_t \notin O_r\}$ . Let  $\hat{P}_x^r$  be the law of  $\tau_r$  under  $\hat{P}_x$ . The following lemma is the same as [9, Lemma 3.2], so the proof is omitted.

**Lemma 16.**  $\hat{P}_x^r([0, t])$  is differentiable in  $t$  on  $(0, \infty)$  and there exists a constant  $\hat{C}_1 > 0$  independent of  $x, r$  and  $t$  such that  $\frac{d}{dt}\hat{P}_x^r([0, t]) \leq \hat{C}_1 t^{-1}$ .

For a closed subset  $A$  of  $\mathbb{R}^d$ , a stopping time  $\sigma_A$  is defined as  $\sigma_A = \inf\{t \geq 0 \mid \omega_t \in A\}$ . For  $r > 0$ , let

$$A_r = \{x \in \mathbb{R}^d \mid \overline{B}(x, r) \cap \Sigma_r \neq \emptyset\}, \quad (13)$$

where  $\Sigma_r$  is defined in (9).

**Proposition 17.** For  $\gamma > 0$ ,  $u > 0$ ,  $r \in (0, \gamma]$ , and  $x \in \overline{O}$ , the inequality

$$\hat{P}_x \left[ 0 \leq \inf_{t \in [0, u]} q(\omega_t) \leq r, \sigma_{A_\gamma} > u \right] \leq \left( \frac{d-1}{\gamma} + \hat{C}_2 u^{-1/2} \right) r \quad (14)$$

holds with  $\hat{C}_2 = 4\hat{C}_1 + 2$ .

**Proof.** Denote  $(d-1)/\gamma$  by  $C'_\gamma$ . If  $x \in A_\gamma$ , (14) is trivial since the left-hand side is 0. Suppose  $x \in \overline{O} \setminus (O_r \cup A_\gamma)$ . Then,  $q(x) < r \leq \gamma$ . Take  $y \in \partial O$  such that  $|x - y|_{\mathbb{R}^d} = q(x)$ . Because  $x \notin A_\gamma$  and  $q(x) < \gamma$ ,  $y$  does not belong to  $\Sigma_\gamma$ : that is,  $\delta(y) \geq \gamma$ . From Proposition 15,

$$\hat{P}_x \left[ 0 \leq \inf_{t \in [0, u]} q(\omega_t) \leq r \right] \leq \left( \frac{d-1}{\delta(y)} + u^{-1/2} \right) q(x) \leq (C'_\gamma + u^{-1/2})r.$$

Next, suppose  $x \in O_r \setminus A_\gamma$ . Then,

$$\begin{aligned} \left\{ 0 \leq \inf_{t \in [0, u]} q(\omega_t) \leq r \right\} &= \left\{ \tau_r \leq u, 0 \leq \inf_{t \in [0, u - \tau_r]} q((\theta_{\tau_r} \omega)_t) \right\} \\ &\subset \bigcup_{k=1}^{\infty} \left\{ 2^{-k}u < u - \tau_r \leq 2^{-k+1}u, 0 \leq \inf_{t \in [0, 2^{-k}u]} q((\theta_{\tau_r} \omega)_t) \right\} \cup \{\tau_r = u\}. \end{aligned}$$

From Lemma 16,  $\hat{P}_x[\tau_r = u] = 0$ . From the strong Markov property and Proposition 15,

$$\begin{aligned} &\hat{P}_x \left[ 2^{-k}u < u - \tau_r \leq 2^{-k+1}u, 0 \leq \inf_{t \in [0, 2^{-k}u]} q((\theta_{\tau_r} \omega)_t), \sigma_{A_\gamma} > u \mid \hat{\mathcal{F}}_{\tau_r} \right] \\ &\leq \mathbf{1}_{\{2^{-k}u < u - \tau_r \leq 2^{-k+1}u\}} \cdot \hat{E}_x \left[ \hat{P}_{\omega_{\tau_r}} \left[ 0 \leq \inf_{t \in [0, 2^{-k}u]} q(\omega_t) \right]; \omega_{\tau_r} \in \partial O_r \setminus A_\gamma \right] \\ &\leq \mathbf{1}_{\{2^{-k}u < u - \tau_r \leq 2^{-k+1}u\}} \cdot (C'_\gamma + (2^{-k}u)^{-1/2})r \\ &\leq \mathbf{1}_{\{2^{-k}u < u - \tau_r \leq 2^{-k+1}u\}} \cdot (C'_\gamma + ((u - \tau_r)/2)^{-1/2})r. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \hat{P}_x \left[ 0 \leq \inf_{t \in [0, u]} q(\omega_t) \leq r, \sigma_{A_\gamma} > u \right] \\
& \leq r \hat{E}_x [C'_\gamma + ((u - \tau_r)/2)^{-1/2}; \tau_r \leq u] \\
& \leq r C'_\gamma + r \left( \frac{u}{4} \right)^{-1/2} \hat{P}_x [\tau_r \leq u/2] + r \int_{u/2}^u \left( \frac{u-s}{2} \right)^{-1/2} \hat{C}_1 s^{-1} ds \\
& \quad (\text{from Lemma 16}) \\
& \leq r C'_\gamma + 2ru^{-1/2} + \frac{2\hat{C}_1 r}{u} \int_{u/2}^u \left( \frac{u-s}{2} \right)^{-1/2} ds \\
& = r C'_\gamma + 2ru^{-1/2} + 4\hat{C}_1 ru^{-1/2}.
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 18.** *Let  $\{B_n\}_{n=1}^\infty$  be a decreasing sequence of closed subsets of  $\mathbb{R}^d$ . If  $\text{Cap}_{d-4}(\bigcap_{n=1}^\infty B_n) = 0$ , then  $\text{Cap}(\Theta_{B_n})$  converges to 0 as  $n \rightarrow \infty$ .*

**Proof.** From (11),  $\text{Cap}(\Theta_{\bigcap_{n=1}^\infty B_n}) = 0$ . For  $\varepsilon > 0$ , take a compact subset  $F$  of  $E$  such that  $\text{Cap}(E \setminus F) < \varepsilon$ . Since  $\{\Theta_{B_n} \cap F\}_{n=1}^\infty$  is a decreasing sequence of compact sets,

$$\lim_{n \rightarrow \infty} \text{Cap}(\Theta_{B_n} \cap F) = \text{Cap}(\Theta_{\bigcap_{n=1}^\infty B_n} \cap F) = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \text{Cap}(\Theta_{B_n}) \leq \lim_{n \rightarrow \infty} \text{Cap}(\Theta_{B_n} \cap F) + \text{Cap}(E \setminus F) \leq \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we obtain the claim.  $\square$

**Proof of Theorem 11.** Recall the definition of  $A_r$  in (13). We note that

$$\bigcap_{k=1}^\infty A_{1/k} = \Sigma. \quad (15)$$

From Proposition 17 and (12),

$$\begin{aligned}
\mu(\Xi_{\overline{O}} \setminus \Xi_O) &= \hat{P}_0 \left[ \inf_{t \in [0, T]} q(\omega_t) = 0 \right] \\
&\leq \lim_{n \rightarrow \infty} \hat{P}_0 \left[ \left\{ \inf_{t \in [0, T]} q(\omega_t) = 0 \right\} \cap \{\sigma_{A_{1/n}} > T\} \right] + \mu(\Theta_\Sigma) \\
&= 0.
\end{aligned}$$



That is,  $\mathbf{1}_{\Xi_{\overline{O}}} = \mathbf{1}_{\Xi_O}$   $\mu$ -a.e. Since  $\mathbf{1}_{\Xi_O}$  is a lower semicontinuous function,  $\mathbf{1}_{\Xi_{\overline{O}}} \in QR$  (cf. [5, p. 230]).

For  $k \in \mathbb{N}$ , let  $B_k$  denote the closure of  $A_{1/k}^c$ , and let  $F_k = \Xi_{B_k}$ . Then,  $\{F_k\}_{k=1}^\infty$  is an  $\mathcal{E}$ -nest from (12), (15), and Lemma 18. Since  $A_{1/(k+1)} \subset B_k^c$ ,  $\Theta_{A_{1/(k+1)}} \subset E \setminus F_k$  for each  $k$ . Let  $h(w) = \inf_{t \in [0, T]} q(w(t))$  for  $w \in E$  and  $\chi(t) = (0 \vee t) \wedge 1$  for  $t \in \mathbb{R}$ . For  $m \in \mathbb{N}$ , let  $f_m(w) = \chi(mh(w))$  for  $w \in E$ . Then,  $f_m \in \mathbb{D}^{1,1}$  and  $f_m$  converges to  $\mathbf{1}_{\Xi_O}$  in  $L^1$  as  $m \rightarrow \infty$ . Moreover, for  $k \in \mathbb{N}$  and  $m \geq k+1$ ,

$$\begin{aligned} \|(\nabla f_m) \cdot \mathbf{1}_{F_k}\|_1 &\leq \int_{F_k} m \cdot \mathbf{1}_{\{0 \leq h \leq 1/m\}} d\mu \\ &\leq m\mu(\{0 \leq h \leq 1/m\} \setminus \Theta_{A_{1/(k+1)}}) \\ &\leq (k+1)(d-1) + \hat{C}_2 T^{-1/2} \quad (\text{from Proposition 17}). \end{aligned}$$

Therefore,  $\sup_{m \in \mathbb{N}} \|(\nabla f_m) \cdot \mathbf{1}_{F_k}\|_1 < \infty$  for each  $k \in \mathbb{N}$ . From Theorem 3, we conclude that  $\mathbf{1}_{\Xi_{\overline{O}}} \in BV_{\text{loc}}$ .

The identity  $\text{Cap}^{\mathbf{1}_{\Xi_{\overline{O}}}}(\partial \Xi_{\overline{O}} \setminus \partial' \Xi_{\overline{O}}) = 0$  is proved in a similar way to the proof of [9, Theorem 2.4]. We provide details for the readers' convenience. Take an  $\mathcal{E}$ -nest  $\{F'_k\}_{k=1}^\infty$  and functions  $\{\varphi_k\}_{k=1}^\infty$  obtained by applying Lemma 4 to the  $\mathcal{E}$ -nest  $\{F_k\}_{k=1}^\infty$  defined above. For  $s \in (0, T)$ , we define a closed set  $V_s$  by

$$V_s = \left\{ w \in E \left| \inf_{t \in [0, s]} q(w(t)) = 0 \text{ and } \inf_{t \in [s, T]} q(w(t)) = 0 \right. \right\}.$$

Then,  $\partial \Xi_{\overline{O}} \setminus \partial' \Xi_{\overline{O}} \subset \bigcup_{s \in (0, T) \cap \mathbb{Q}} V_s$ . Fix  $s \in (0, T)$  and define a map  $f: E \rightarrow \mathbb{R}^2$  by

$$f(w) = \left( \inf_{t \in [0, s]} q(w(t)), \inf_{t \in [s, T]} q(w(t)) \right).$$

Fix  $k \in \mathbb{N}$  and take  $l \in \mathbb{N}$  such that  $\varphi_k \in \mathbb{D}_{F_l}^{1,2}$ . Let  $\varepsilon > 0$ . Henceforth,  $C$  denotes an unimportant positive constant independent of  $\varepsilon$  that may vary line by line. Take a smooth function  $g$  on  $[0, \infty)$  such that

$$g(t) = \begin{cases} 1 & t \in [0, e^{-2/\varepsilon}], \\ -3\varepsilon \log t - 4 & t \in [e^{-14/(9\varepsilon)}, e^{-13/(9\varepsilon)}], \\ 0 & t \in [e^{-1/\varepsilon}, \infty), \end{cases}$$

and  $-3\varepsilon/t \leq g'(t) \leq 0$  for all  $t > 0$ . We define a function  $\zeta: \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $\zeta(x, y) = \sqrt{x^2 + y^2}$  and set  $\iota = g \circ \zeta$ . Since  $\iota \circ f$  is a bounded  $H$ -Lipschitz

continuous function, it belongs to  $\mathbb{D}_b^{1,2}$ . Let  $\eta = (\iota \circ f)\varphi_k + e_{E \setminus F'_k} \in \mathbb{D}_b^{1,2}$ . Then,  $\eta \geq 1$   $\mu$ -a.e. on some open set including  $V_s$ . Since  $(\iota \circ f)\varphi_k \in \mathbb{D}_{F_l}^{1,2}$ ,

$$\begin{aligned} & \mathcal{E}^{1 \setminus \overline{\sigma}}(\eta|_{\Xi_{\overline{\sigma}}}, \eta|_{\Xi_{\overline{\sigma}}}) \\ & \leq 2\mathcal{E}^{1 \setminus \overline{\sigma}}((\iota \circ f)\varphi_k|_{\Xi_{\overline{\sigma}}}, (\iota \circ f)\varphi_k|_{\Xi_{\overline{\sigma}}}) + 2\mathcal{E}^{1 \setminus \overline{\sigma}}(e_{E \setminus F'_k}|_{\Xi_{\overline{\sigma}}}, e_{E \setminus F'_k}|_{\Xi_{\overline{\sigma}}}) \\ & \leq 2 \int_{\Xi_{\overline{\sigma}} \cap F_l} |\nabla(\iota \circ f)|_H^2 d\mu + 4\mathcal{E}(\varphi_k, \varphi_k) + 2\mathcal{E}(e_{E \setminus F'_k}, e_{E \setminus F'_k}). \end{aligned}$$

Denoting the gradient operator on  $\mathbb{R}^2$  by  $\nabla_{\mathbb{R}^2}$ , we have

$$\begin{aligned} \int_{\Xi_{\overline{\sigma}} \cap F_l} |\nabla(\iota \circ f)|_H^2 d\mu &= \int_{\Xi_{\overline{\sigma}} \cap B_l} |\langle (\nabla f)(w), (\nabla_{\mathbb{R}^2} \iota)(f(w)) \rangle_{\mathbb{R}^2}|_H^2 \mu(dw) \\ &\leq C \int_{\Xi_{\overline{\sigma}} \cap B_l} |(\nabla_{\mathbb{R}^2} \iota)(f(w))|_{\mathbb{R}^2}^2 \mu(dw) \\ &= C \int_{\{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}} |\nabla_{\mathbb{R}^2} \iota|_{\mathbb{R}^2}^2 d(f_*(\mu|_{\Xi_{\overline{\sigma}} \cap B_l})) \\ &=: I_1. \end{aligned}$$

In the first line,  $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$  denotes a pairing between elements in  $H \otimes \mathbb{R}^2$  and in  $\mathbb{R}^2$  that takes values in  $H$ . We note that

$$\begin{aligned} |\nabla_{\mathbb{R}^2} \iota|_{\mathbb{R}^2}^2 &= (\partial \iota / \partial x)^2 + (\partial \iota / \partial y)^2 \\ &= (g' \circ \zeta(x, y))^2 \frac{x^2}{x^2 + y^2} + (g' \circ \zeta(x, y))^2 \frac{y^2}{x^2 + y^2} \\ &= (g' \circ \zeta(x, y))^2. \end{aligned}$$

By letting  $\psi = (\zeta \circ f)_*(\mu|_{\Xi_{\overline{\sigma}} \cap B_l})$ , we obtain

$$I_1 = C \int_0^\infty g'(r)^2 \psi(dr) \leq 9C\varepsilon^2 \int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} r^{-2} \psi(dr) =: I_2.$$

We now have

$$\begin{aligned} \Psi(r) &:= \psi([0, r]) = (f_*(\mu|_{\Xi_{\overline{\sigma}} \cap B_l}))(\zeta^{-1}([0, r])) \\ &= \mu \left[ \left\{ w \in \Xi_{\overline{\sigma}} \cap B_l \mid \inf_{t \in [0, s]} q(w(t))^2 + \inf_{t \in [s, T]} q(w(t))^2 \leq r^2 \right\} \right] \\ &\leq \mu \left[ \left\{ w \in \Xi_{\overline{\sigma} \setminus A_{1/(l+1)}} \mid \begin{array}{l} 0 \leq \inf_{t \in [0, s]} q(w(t)) \leq r, \\ 0 \leq \inf_{t \in [s, T]} q(w(t)) \leq r \end{array} \right\} \right] \end{aligned}$$

$$\leq \hat{E}_0 \left[ \hat{P}_{\omega_s} \left[ 0 \leq \inf_{t \in [0, T-s]} q(\omega_t) \leq r, \sigma_{A_1/(l+1)} > T-s \right]; \right. \\ \left. 0 \leq \inf_{t \in [0, s]} q(\omega_t) \leq r, \sigma_{A_1/(l+1)} > s \right].$$

From Proposition 17, if  $r \leq 1/(l+1)$ , this expectation is dominated by

$$((d-1)(l+1) + C(T-s)^{-1/2})r\hat{P}_0 \left[ 0 \leq \inf_{t \in [0, s]} q(\omega_t) \leq r, \sigma_{A_1/(l+1)} > s \right] \\ \leq ((d-1)(l+1) + C(T-s)^{-1/2})((d-1)(l+1) + Cs^{-1/2})r^2 \\ = Cr^2.$$

Thus, for  $\varepsilon \leq 1/\log(l+1)$ ,

$$I_2 = 9C\varepsilon^2 \int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} r^{-2} d\Psi(r) \\ = C\varepsilon^2 \left\{ \left[ \frac{\Psi(r)}{r^2} \right]_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} + \int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} \frac{2\Psi(r)}{r^3} dr \right\} \\ \leq C\varepsilon^2 \left( C + \int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} \frac{C}{r} dr \right) = C\varepsilon^2(1 + 1/\varepsilon).$$

Therefore,

$$\text{Cap}^{1\Xi\overline{\sigma}}(V_s) \leq \mathcal{E}^{1\Xi\overline{\sigma}}(\eta|_{\Xi\overline{\sigma}}, \eta|_{\Xi\overline{\sigma}}) + \|\eta|_{\Xi\overline{\sigma}}\|_{L^2(\mu|_{\Xi\overline{\sigma}})}^2 \\ \leq 2 \int_{\Xi\overline{\sigma} \cap F_l} |\nabla(\iota \circ f)|_H^2 d\mu + 4\mathcal{E}(\varphi_k, \varphi_k) + 2\mathcal{E}(e_{E \setminus F'_k}, e_{E \setminus F'_k}) \\ + 2\Psi(e^{-1/\varepsilon}) + 2\|e_{E \setminus F'_k}\|_2^2 \\ \leq C\varepsilon^2(1 + 1/\varepsilon) + 4\mathcal{E}(\varphi_k, \varphi_k) + 2\text{Cap}(E \setminus F'_k) + Ce^{-2/\varepsilon}.$$

By letting  $\varepsilon \rightarrow 0$  and then  $k \rightarrow \infty$ , we obtain  $\text{Cap}^{1\Xi\overline{\sigma}}(V_s) = 0$ . Therefore,

$$\text{Cap}^{1\Xi\overline{\sigma}}(\partial\Xi\overline{\sigma} \setminus \partial'\Xi\overline{\sigma}) \leq \sum_{s \in (0, T) \cap \mathbb{Q}} \text{Cap}^{1\Xi\overline{\sigma}}(V_s) = 0. \quad \square$$

**Remark 19.** In this section, we considered only one-sided pinned path spaces as the underlying space for simplicity. However, the general idea of the argument is also valid for pinned path spaces, as discussed in [9].

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